Image inpainting through anisotropic diffusion on biologically inspired manifolds

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Abstract

In mammals **image inpainting** is intrinsically performed at the level of the **Primary Visual Cortex V1** in order to fill the gaps in the field of view and make sense of the surroundings when obstructions are present. It is possible to model this part of the brain as $SE(2)$, a **sub-Riemannian manifold**, where the neural activity is propagated through **anisotropic diffusion**. Solving the diffusion equation yields image restoration.

Milestones of this approach are the papers from Citti and Sarti [1] and Boscain et al. [2].

In this work the aim is to propose a new biologically supported lift operator (**Gaussian lift**) and a novel versatile 2-steps technique for image restoration (**WaxOn-WaxOff**).

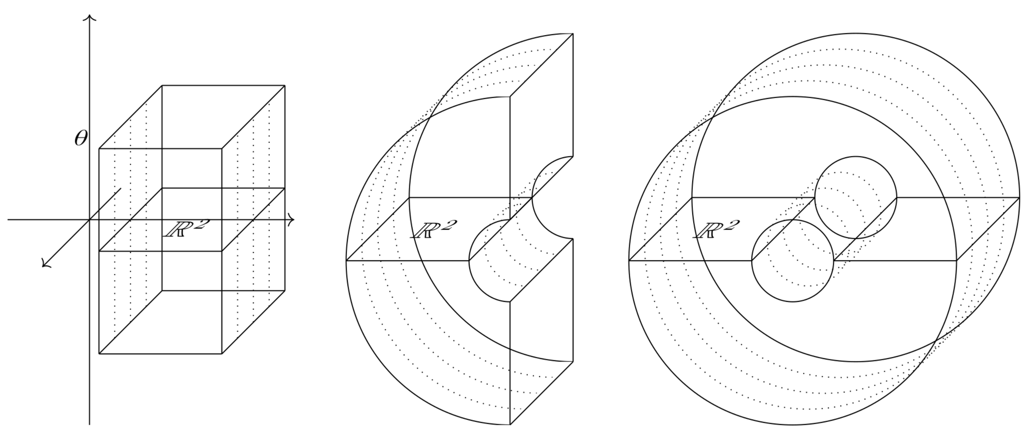
The Primary Visual Cortex V1

Neurons are grouped into **simple cells** that are sensitive to intensity and orientation at a certain position on the retina. Simple cells sensitive to a certain postion, regardless of orientations, are grouped into **hypercolumns**. Excitatory synapses are present between simple cells with similar orientations located in the same hypercolumn and simple cells with same orientation located in spatially close hypercolumns. Synapses in the same hypercolumn are said to be **vertical** while synapses between simple cells in different hypercolumns with same orientation are said to be **horizontal**.

The manifold $SE(2)$

$$SE(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \middle| x, y \in \mathbb{R},\ \theta \in [0, 2\pi) \right\}$$

$$SE(2) \simeq \mathbb{R}^2 \times S^1$$



Define the following basis of left invariant vector fields by left translation:

$$\begin{aligned} \partial_x|_{\mathbb{1}} &\to X_1 = \cos\theta\,\partial_x + \sin\theta\,\partial_y \\ \partial_\theta|_{\mathbb{1}} &\to X_2 = \partial_\theta \\ \partial_y|_{\mathbb{1}} &\to X_3 = -\sin\theta\,\partial_x + \cos\theta\,\partial_y \end{aligned}$$

Choosing $\mathcal{H} = \{X_1, X_2\}$ as horizontal distribution makes $SE(2)$ into a sub-Riemannian geometry.
X_1 corresponds to the direction of the horizontal synapses while X_2 to the direction of vertical synapses.

The restoration algorithm

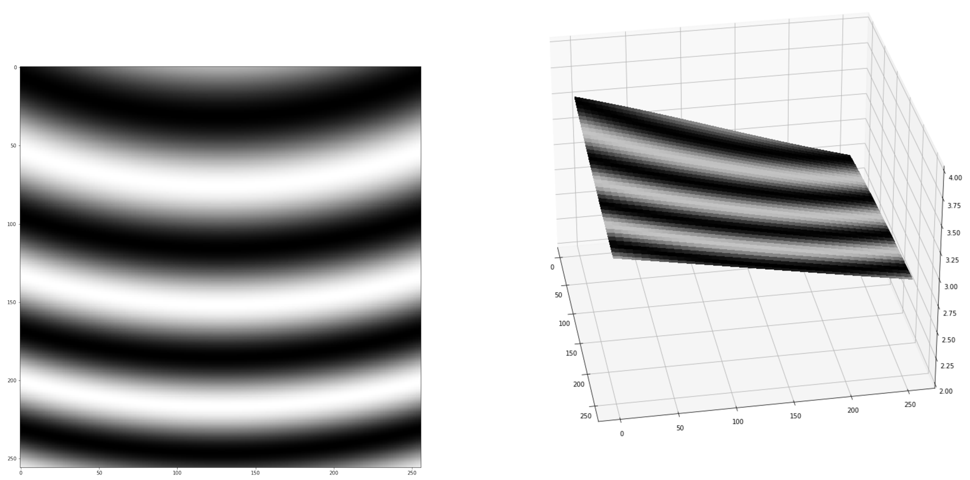
The algorithm is **blind** as it is not known which pixels are corrupted and which are not. Image restoration is achieved by following 4 steps, starting with a grayscale digital image.

1. **Smooth** the image with a Gaussian kernel (guarantees the image is generically a Morse function)
2. **Lift** $\mathbb{R}^2 \to SE(2)$
3. Apply anisotropic **diffusion** over $SE(2)$
4. **Project** $SE(2) \to \mathbb{R}^2$

The Citti and Sarti lift $\mathbb{R}^2 \to SE(2)$

For a Morse function lift the image to the surface in $SE(2)$ such that all the points are lifted corresponding to the orientation of the **level lines** at that point.

$$\tilde{I}(x, y, \theta) := \begin{cases} I(x, y) & |X_3(\theta)| = \max_{\bar{\theta}} |X_3(\bar{\theta})| \\ 0 & \text{otherwise} \end{cases}$$



(a) Original image (b) Lifted image according to the direction of the level lines

The diffusion equation on $SE(2)$

$$\begin{cases} \partial_t u = \Delta_H u \\ u(0, x, y, \theta) = \mathcal{L}(I(x, y, \theta)) \end{cases}$$
$$\Delta_H = X_1^2 + \beta^2 X_2^2$$

where β expresses the strength of the vertical synapses with respect to the horizontal ones.

This can be brought to Fourier space with respect to the spatial variable to obtain

$$\begin{cases} \partial_t \tilde{u} = \beta^2 \partial_\theta^2 \tilde{u} - 4\pi^2 (x\cos\theta + y\sin\theta)^2 \tilde{u} \\ \tilde{u}(0, \mathbf{x}, \theta) = \tilde{\mathcal{L}}(I)(\mathbf{x}, \theta) \end{cases}$$

which is highly parallelizable.

Gaussian lift

From biology (Marcelja [3], Jones and Palmers [4]) we know that simple cells act like **Gabor Filters**. In particular the activity level **decays exponentially** the more the orientation of the simple cells differ from the orientation perceived. This effect is not taken into account by the Citti and Sarti lifting procedure.

A **new lift operator** that follows such assumption can be defined as

$$\mathcal{L}(I)(x, y, \theta) = I(x, y) \cdot \exp\left(- \frac{\left\langle \frac{\nabla I(x,y)}{|\nabla I(x,y)|}, (\cos\theta, \sin\theta) \right\rangle^2}{2\sigma^2} \right)$$

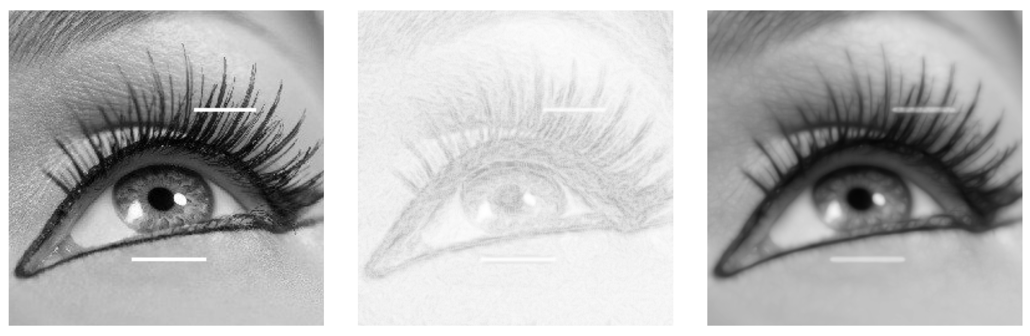
with inverse

$$\pi(\tilde{I})(x, y) = \frac{1}{\pi} \int_0^\pi \tilde{I}(x, y, \theta) \cdot e^{-\mu(x,y,\theta)} d\theta$$

$$\mu(x, y, \theta) := \min_{\tilde{\theta} \in [0,\pi]} Q(x, y, \tilde{\theta}) - Q(x, y, \theta)$$

$$Q(x, y, \theta) := -\int_0^\theta \frac{\partial}{\partial \tilde{\theta}} \ln(\tilde{I}(x, y, \tilde{\theta})) d\tilde{\theta}$$

Empirical results show that this lifting operator is **robust** when diffusion is applied and is extremely **effective** at image restoration.



(a) Original image (b) Diffusion over Citti and Sarti lift $T = 3$ (c) Diffusion over Gaussian lift $T = 3$

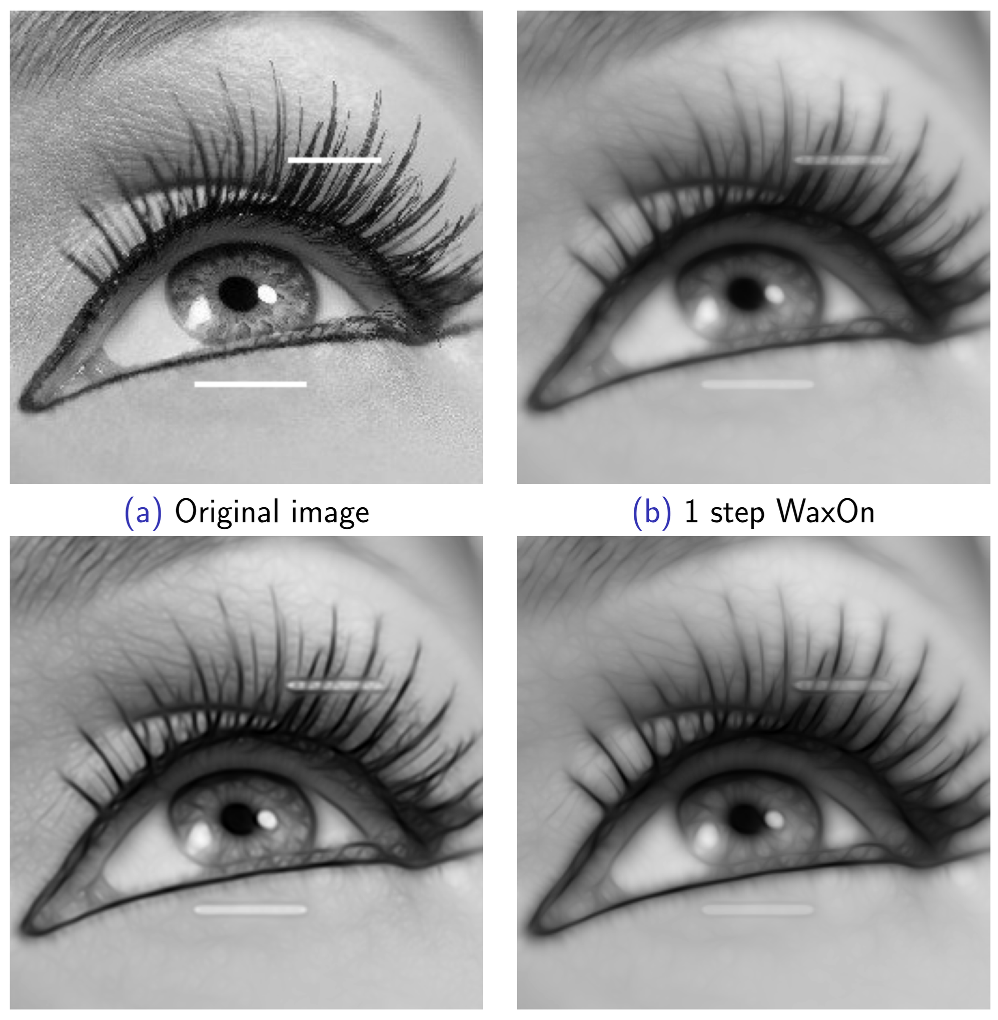
Figure: The original image (a) is lifted with Citt-Sarti lift and diffusion is applied (b). The same image is lifted with Gaussian lift and the same diffusion is applied (c). Gaussian lift yields a more stable approach.

WaxOn - WaxOff

"Wax on, left hand. Wax off, right hand. Wax on, wax off. [...] Don't forget to breath....very important!"

(Mr. Miyagi, The Karate Kid)

If the diffusion equation is reversed, for small values of T it is possible to recover the initial heat profile over $SE(2)$ obtaining a **perception-based sharpening filter**. We can exploit this information to produce a 2-step procedure where we first diffuse with a small β and then reverse the diffusion with a large β.



(a) Original image (b) 1 step WaxOn (c) 1 step WaxOn-WaxOff (d) 2 steps WaxOn-WaxOff

Figure: The original image (a) is lifted with Gaussian lift and diffusion *WaxOn* is applied (b) followed by regression *WaxOff* for small T (c). Multiple iterations of WaxOn-WaxOff are sequentially applied to produce (d)

By applying the two steps **repeatedly** the WaxOn portion of the algorithm yields restoration while the WaxOff portion sharpens the result while retaining restored information. Together with the Gaussian lift the resulting method is robust and produces extremely sharp restoration.

References

[1]: G. Citti and A. Sarti. "A cortical based model of perceptual completion in the roto-translation space", J. Math. Imaging Vis. (2006)
[2]: U. Boscain et al. "Anthropomorphic image inpainting through hypoelliptic diffusion", JSIAM j. control optim. (2012)
[3]: S. Marcelja. "Mathematical description of the responses of simple cortical cells". en. In: J. Opt. Soc. Am. (1980)
[4]: J. P. Jones and L. A. Palmer. "An evaluation of the two-dimensional Gabor filter model of simple receptive fields in cat striate cortex". en. In: J. Neurophysiol. (1987)