NCM29

Geometry of the visual cortex with applications to image inpainting and enhancement

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Objective: image processing

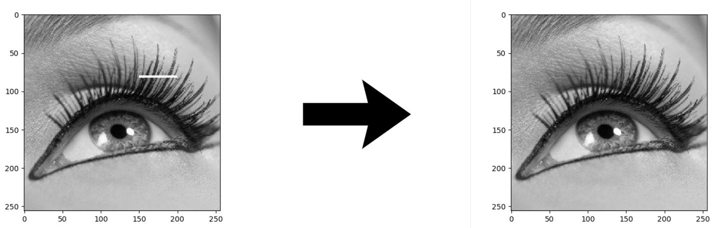


Figure: Image restoration

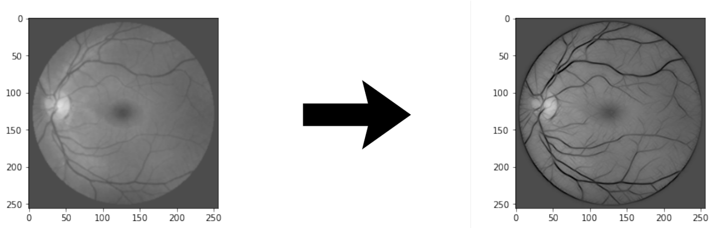
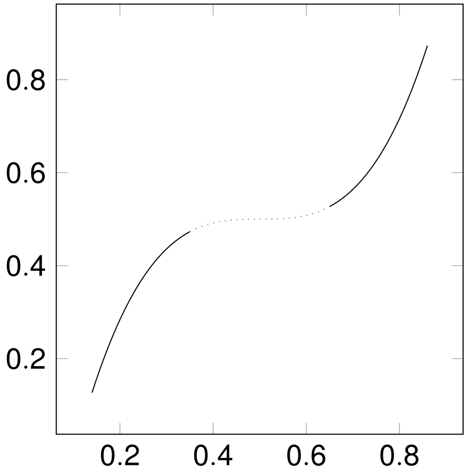


Figure: Image enhancement

Curve completion in 2D



Curve completion in 2D

- $\gamma_0 : [a, b] \cup [c, d] \to \mathbb{R}^2$ a smooth curve that is partially hidden in the interval $t \in (b, c)$.
- We want to find a curve $\gamma : [b, c] \to \mathbb{R}^2$ that completes γ_0 while minimizing a cost $J[\gamma]$.
- Constraints on position: $\gamma(b) = \gamma_0(b)$, $\gamma(c) = \gamma_0(c)$
- Constraints on orientation:
  - $\dot{\gamma}(b) \sim \dot{\gamma}_0(b)$, $\dot{\gamma}(c) \sim \dot{\gamma}_0(c)$
    or
  - $\dot{\gamma}(b) \approx \dot{\gamma}_0(b)$, $\dot{\gamma}(c) \approx \dot{\gamma}_0(c)$
- $J_\beta[\gamma] = \int_b^c \sqrt{\|\dot{\gamma}(t)\|^2 + \beta^2 \|\dot{\gamma}(t)\|^2 K_\gamma^2(t)} dt$

The visual cortex V1

- The **visual cortex V1** is the main region of a mammal's brain for **processing vision.**
- Composed of simple cells that are sensitive to **position** and **orientation.**
- Simple cells that receive enough stimulus from an external source **spike.**
- Simple cells that spike stimulate other simple cells with **same position but different orientation**, and simple cells with **same orientation and close position.**

The visual cortex V1

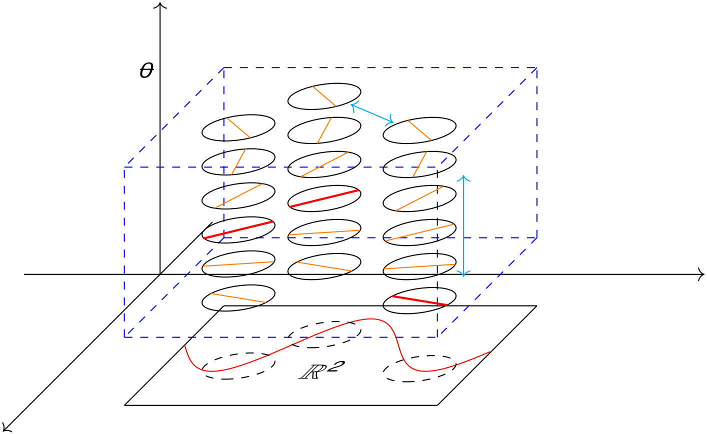


Figure: Hypercolumns of the Visual Cortex V1 under a stimulus (red curve)

The visual cortex V1

$$SE(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; \begin{matrix} x, y \in \mathbb{R}, \\ \theta \in \mathbb{R}/2\pi\mathbb{Z} \end{matrix} \right\}$$

Using coordinates (x, y, θ) we see that $SE(2)$ as a space can be identified with the 3-dimensional cylinder $\mathbb{R}^2 \times S^1$.

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \qquad X_2 = \partial_\theta, \qquad X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$$

$$[X_1, X_2] = -X_3, \qquad [X_2, X_3] = X_1, \qquad [X_1, X_3] = 0.$$

Let $\mathcal{H} = \{X_1, X_2\}$ be the horizontal distribution, then $SE(2)$ endowed with $\mathcal{H}$ is a **bracket-generating sub-Riemannian manifold**.

Lift of curve from $\mathbb{R}^2$ to $SE(2)/\sim$

Consider $SE(2)/\sim$, where $(x, y, \theta) \sim (x, y, \theta + \pi)$.

A curve is lifted from $\mathbb{R}^2$ to $SE(2)/\sim$ by adding a coordinate $\theta \in [0, \pi)$ corresponding to the angle between the orientation of the curve and the horizontal axis $y = 0$, measured counterclockwise.

A lifted curve is projected to $\mathbb{R}^2$ simply by suppressing the third coordinate corresponding to orientation.

Curve completion in $SE(2)/\sim$

Completing a curve in $\mathbb{R}^2$ is equivalent to finding a minimizer in $SE(2)/\sim$, where $(x, y, \theta) \sim (x, y, \theta + \pi)$.

Proposition (Boscain, Charlot, Rossi - 2010)

For all boundary conditions $\gamma_0(b), \gamma_0(c) \in \mathbb{R}^2$ with $\gamma_0(b) \neq \gamma_0(c)$ and $\dot{\gamma}_0(b), \dot{\gamma}_0(c) \in \mathbb{R}^2 \setminus \{0\}$ the cost $J_\beta[\gamma]$ admits a minimizer over the set

$$\mathcal{D} = \left\{ \gamma \in C^2([b, c], \mathbb{R}^2) \middle| \begin{array}{l} \|\dot{\gamma}\|^2 + \|\dot{\gamma}\|^2 K_\gamma^2 \in L^1([b, c], \mathbb{R}) \\ \gamma(b) = \gamma_0(b), \gamma(c) = \gamma_0(c) \\ \dot{\gamma}(b) \approx \dot{\gamma}_0(b), \dot{\gamma}(c) \approx \dot{\gamma}_0(c) \end{array} \right\}$$

Beyond curve completion: image diffusion

- Not working with a single curve but with **many level curves.**
- Consider all possible admissible paths and model the controls by independent **Wiener processes** u_t and v_t obtaining the following SDE:

$$\begin{pmatrix} dx_t \\ dy_t \\ d\theta_t \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos\theta_t \\ \sin\theta_t \\ 0 \end{pmatrix} \circ du_t + \sqrt{2\beta} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \circ dv_t$$

- The diffusion process associated to such SDE is

$$\frac{\partial \Psi}{\partial_t} = \Delta \Psi$$

where

$$\Delta = X_1^2 + \beta^2 X_2^2 = \left(\cos\theta \frac{\partial}{\partial_x} + \sin\theta \frac{\partial}{\partial y}\right)^2 + \beta^2 \frac{\partial^2}{\partial \theta^2}.$$

The Citti-Sarti-Boscain algorithm

1. **Smooth** the image by convolution with a Gaussian kernel (guarantees the image is generically a Morse function)

2. **Lift** $\mathbb{R}^2 \to SE(2)$

3. Solve the Cauchy problem

$$\Delta_\beta = X_1^2 + \beta^2 X_2^2$$

$$\begin{cases} \partial_t u = \Delta_\beta u, \\ u(0, x, y, \theta) = \tilde{I}(x, y, \theta) \end{cases}$$

   over $SE(2)$

4. **Project** $SE(2) \to \mathbb{R}^2$

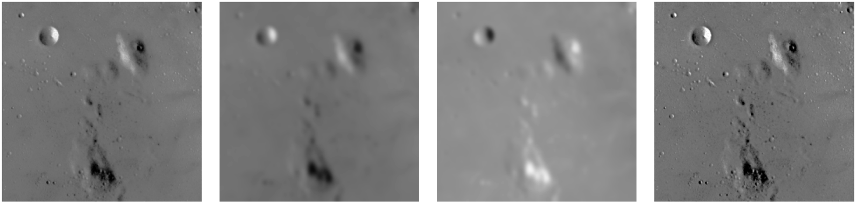
Some examples

(a) Original image (b) $T = 21$ with $\beta = 0.1$ (c) $T = 21$ with $\beta = 1$ (d) $T = 21$, $\beta = 10$

Figure: An image (a) is processed with the Citti-Sarti-Boscain algorithm for different values β (b,c,d)

How to deal with blur: unsharp filtering

$$I \mapsto I + C(I - I * G_\sigma)$$



(a) Original image

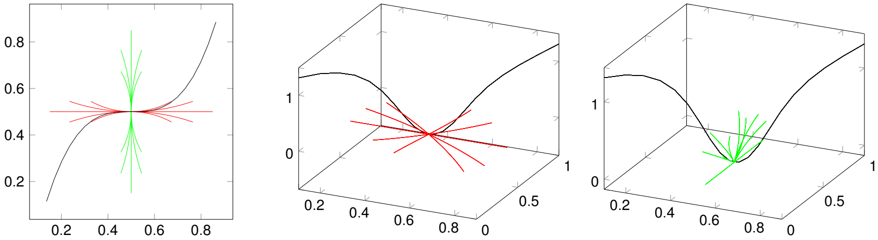
(b) Blurred image with $\sigma = 5$

(c) Negative of the blurred image

(d) Sharpened image with $C = 1$

Figure: Example of usage of the unsharp filter applied to a low-contrast image of the surface of the moon

X_1, X_2, and X_3



(a) Projections to $\mathbb{R}^2$ of the integral lines

(b) Integral lines of $X_1^2 + \beta^2 X_2^2$

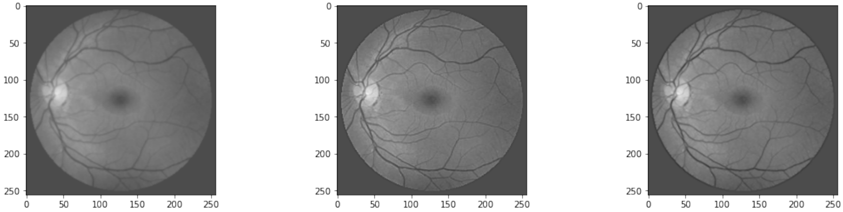
(c) Integral lines of $X_3^2 + \beta^2 X_2^2$

Figure: Integral lines of the vector fields $X_1^2 + \beta^2 X_2^2$ (red) and $X_3^2 + \beta^2 X_2^2$ (green) for a polynomial curve, at point $\left(\frac{1}{2}, \frac{1}{2}\right)$, varying the coefficient β.

Unsharp filtering on $SE(2)/\sim$

The **undesired blurring** is obtained from the Cauchy problem

$$\begin{cases} \partial_t u = \Delta_\beta u, \\ u(0,x,y,\theta) = \tilde{I}(x,y,\theta) \end{cases} \qquad \Delta_\beta = X_3^2 + \beta^2 X_2^2$$



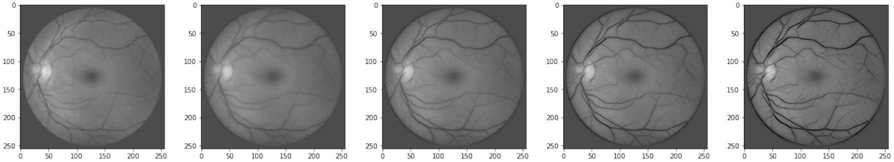
(a) Original image

(b) $\mathbb{R}^2$ unsharp filter

(c) $SE(2)$ unsharp filter

Figure: Retinal image (a) sharpened using the classical unsharp filter over $\mathbb{R}^2$ (b) and using the proposed sharpening method (c).

Retinal image enhancement through unsharp filtering



(a) Original image (b) $C = 0.5$ (c) $C = 1$ (d) $C = 1.5$ (e) $C = 2$

Figure: The original image (a) is processed with diffusion under $\Delta_\beta = X_1^2 + \beta^2 X_2^2$ and sharpended with varying coefficients C (b,c,d,e)

WaxOn-WaxOff

- WaxOn: diffusion problem with $\Delta_\beta = X_1^2 + \beta^2 X_2^2$
- WaxOff: inverse problem with $\Delta_\beta = X_3^2 + \beta^2 X_2^2$

(a) Original image

(b) $X_1^2 + \beta^2 X_2^2$ diffusion

(c) 1 cycle of WaxOn-WaxOff

(d) 3 cycles of WaxOn-WaxOff

Figure: From (a) the CSB algorithm is applied to obtain (b). One iteration of WaxOn-WaxOff for small T_2 produces (c) while multiple iterations of WaxOn-WaxOff are sequentially applied to produce (d). Total diffusion time in (b) and (d) is the same.

References

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'Existence of planar curves minimizing length and curvature', 2010
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Geometry of the visual cortex with applications to image inpainting and enhancement, 2023
TBA

sR manifold

Definition

A sub-Riemannian manifold is a triplet $(M, \mathcal{H}, g)$ with M being a connected manifold, $\mathcal{H} \subset TM$ a linear subbundle and $g = \langle \cdot, \cdot \rangle$ a fiber-metric defined on on the subbundle .

We call $\mathcal{H} \subset TM$ in this definition the *horizontal distribution*. A sub-Riemannian manifold can be considered as a limiting case of a Riemannian manifold where the distances of vectors outside of approach infinity. Curves $\gamma : [a, b] \to M$ with a finite length will then need to be *a horizontal curve*: an absolutely continuous curve satisfying $\dot{\gamma}(t) \in_{\gamma(t)}$ for almost every t. For such a curve, we can define its length by

$$(\gamma) = \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle^{1/2} \, dt.$$

sR distance

We can then also introduce the corresponding sub-Riemannian distance by

$$d_g(x,y) = \inf \left\{ (\gamma) \ : \ \begin{array}{c} \gamma : [a,b] \to M \text{ horizontal} \\ \gamma(a) = x, \gamma(b) = y \end{array} \right\}.$$

In general, there might not be any curve connecting a point x and y, meaning that the distance above will be infinite. It is therefore typical to require the horizontal bundle sub-Riemannian manifold to be bracket-generating

Bracket generating distribution

$$\hat{\mathfrak{X}} = \left\{ [X_{i_1}, [X_{i_2}, [\cdots [X_{l=1}, X_l]] \cdots]] \mid X_{i_j} \in, l = 1, 2, 3, \ldots, \right\},$$

where we interpret the case $l = 1$ simply as the vector field X_{i_1} itself. We then make the following definition.

Definition

We say that is bracket-generating if for every $x \in M$,

$$T_x M = \{X(x) : X \in \hat{\mathfrak{X}}\}.$$

Sub-Laplacian

Consider a second order operator L on a manifold M, which in local coordinates

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j},$$

with $(a_{ij}(x))$ being positive semi-definite with a constant rank k. Such an operator can locally be written as $L = \sum_{=1}^{k} X_k^2 + X_0$. Define a sR structure on $(, g)$ on M by making $X_1, \ldots, X_k$ into a local orthonormal basis. If L is required to be symmetric, i.e. $\int_M f_1(Lf_2)d\mu = \int_M f_2(Lf_1)d\mu$ for any pair of smooth functions $f_1, f_2 \in C_0^\infty(M)$ of compact support, then the *symmetric* operator is unique with respect to a given volume density $d\mu$. We call this operators *the sub-Laplacian of* $(M, , g)$ *and* $d\mu$.

Hypoellipticity of Δ_β

Theorem

Let L be the sub-Laplacian of a sub-Riemannian structure $(M, , g)$ with volume element $d\mu$. Assume that is bracket-generating. Then L and the heat operator $\partial_t - L$ are hypoelliptic. Furthermore, for the heat-semigroup e^{tL}, we have density

$$e^{tL}f(x) = \int_M p_t(x, y)f(y)\, d\mu,$$

where $p_t(x, y)$ is a smooth, strickly positive function that is symmetric in x and y. Furthermore, we have short time asymptotics

$$\lim_{t\downarrow 0} 2t \log p_t(x, y) = d_g(x, y).$$

Lift of an image